Existence of zeros for operators taking their values in the dual of a Banach space

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Throughout the sequel, E denotes a reflexive real Banach space and E^* its topological dual. We also assume that E is locally uniformly convex. This means that for each $x \in E$, with $\|x\| = 1$, and each $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y \in E$ satisfying $\|y\| = 1$ and $\|x - y\| \geq \epsilon$, one has $\|x + y\| \leq 2(1 - \delta)$. Recall that any reflexive Banach space admits an equivalent norm with which it is locally uniformly convex ([1], p. 289). For $r > 0$, we set $B_r = \{x \in E : \|x\| \leq r\}$.

Moreover, we fix a topology τ on E, weaker than the strong topology and stronger than the weak topology, such that (E, τ) is a Hausdorff locally convex topological vector space with the property that the τ-closed convex hull of any τ-compact subset of E is still τ-compact and the relativization of τ to B_1 is metrizable by a complete metric. In practice, the most usual choice of τ is either the strong topology or the weak topology provided E is also separable.

The aim of this short paper is to establish the following result and present some of its consequences:

THEOREM 1. - *Let X be a paracompact topological space and $A : X \to E^*$ a weakly continuous operator. Assume that there exist a number $r > 0$, a continuous function $\alpha : X \to \mathbf{R}$ satsfying*

$$|\alpha(x)| \leq r\|A(x)\|_{E^*}$$

for all $x \in X$, a closed set $C \subset X$, and a τ-continuous function $g : C \to B_r$ satisfying

$$A(x)(g(x)) = \alpha(x)$$

for all $x \in C$, in such a way that, for every τ-continuous function $\psi : X \to B_r$ satisfying $\psi_{|C} = g$, there exists $x_0 \in X$ such that

$$A(x_0)(\psi(x_0)) \neq \alpha(x_0) \ .$$

Then, there exists $x^* \in X$ such that $A(x^*) = 0$.

For the reader's convenience, we recall that a multifunction $F : S \to 2^V$, between topological spaces, is said to be lower semicontinuous at $s_0 \in S$ if, for every open set $\Omega \subseteq V$ meeting $F(s_0)$, there is a neighbourhood U of s_0 such that $F(s) \cap \Omega \neq \emptyset$ for all $s \in U$. F is said to be lower semicontinuous if it is so at each point of S.

The following well-known results will be our main tools.

THEOREM A ([3]). - *Let X be a paracompact topological space and $F : X \to 2^{B_1}$ a τ-lower semicontinuous multifunction with nonempty τ-closed convex values.*

Then, for each closed set $C \subset X$ and each τ-continuous function $g : C \to B_1$ satisfying $g(x) \in F(x)$ for all $x \in C$, there exists a τ-continuous function $\psi : X \to B_1$ such that $\psi_{|C} = g$ and $\psi(x) \in F(x)$ for all $\in X$.

THEOREM B ([4]). - *Let X, Y be two topological spaces, with Y connected and locally connected, and let $f : X \times Y \to \mathbf{R}$ be a function satisfying the following conditions:*
(a) for each $x \in X$, the function $f(x, \cdot)$ is continuous, changes sign in Y and is identically zero in no nonempty open subset of Y ;
(b) the set $\{(y, z) \in Y \times Y : \{x \in X : f(x, y) < 0 < f(x, z)\}$ is open in $X\}$ is dense in $Y \times Y$.

Then, the multifunction $x \to \{y \in Y : f(x, y) = 0$ and y is not a local extremum for $f(x, \cdot)\}$ is lower semicontinuous, and its values are nonempty and closed.

Proof of Theorem 1. Arguing by contradiction, assume that $A(x) \neq 0$ for all $x \in X$. For each $x \in X$, $y \in B_1$, put

$$f(x, y) = A(x)(y) - \frac{\alpha(x)}{r}$$

and

$$F(x) = \{z \in B_1 : f(x, z) = 0\} .$$

Also, set

$$X_0 = \{x \in X : |\alpha(x)| < r\|A(x)\|_{E^*}\} .$$

Since A is weakly continuous, the function $x \to \|A(x)\|_{E^*}$, as supremum of a family of continuous functions, is lower semicontinuous. From this, it follows that the set X_0 is open. For each $x \in X_0$, the function $f(x, \cdot)$ is continuous and has no local, nonabsolute, extrema, being affine. Moreover, it changes sign in B_1 since $A(x)(B_1) = [-\|A(x)\|_{E^*}, \|A(x)\|_{E^*}]$ (recall that E is reflexive). Since $f(\cdot, y)$ is continuous for all $y \in B_1$, we then realize that the restriction of f to $X_0 \times B_1$ satisfies the hypotheses of Theorem B, B_1 being considered with the relativization of the strong topology. Hence, the multifunction $F_{|X_0}$ is lower semicontinuous. Consequently, since X_0 is open, the multifunction F is lower semicontinuous at each point of X_0. Now, fix $x_0 \in X \setminus X_0$. So, $|\alpha(x_0)| = r\|A(x_0)\|_{E^*}$. Let $y_0 \in F(x_0)$ and $\epsilon > 0$. Clearly, since y_0 is an absolute extremum of $A(x_0)$ in B_1, one has $\|y_0\| = 1$. Choose $\delta > 0$ so that, for each $y \in E$ satisfying $\|y\| = 1$ and $\|y - y_0\| \geq \epsilon$, one has $\|y + y_0\| \leq 2(1 - \delta)$. By semicontinuity, the function $x \to (\|A(x)\|_{E^*})^{-1}$ is bounded in some neighbourhood of x_0, and so, since the functions α and $A(\cdot)(y_0)$ are continuous, it follows that

$$\lim_{x \to x_0} \frac{\left|A(x)(y_0) - \frac{\alpha(x)}{r}\right|}{\|A(x)\|_{E^*}} = 0 .$$

So, there is a neighbourhood U of x_0 such that

$$\frac{\left|A(x)(y_0) - \frac{\alpha(x)}{r}\right|}{\|A(x)\|_{E^*}} < \frac{\epsilon\delta}{2} \tag{1}$$

for all $x \in U$. Fix $x \in U$. Pick $z \in E$, with $\|z\| = 1$, in such a way that $|A(x)(z)| = \|A(x)\|_{E^*}$ and

$$\left(A(x)(z) - \frac{\alpha(x)}{r}\right)\left(A(x)(y_0) - \frac{\alpha(x)}{r}\right) \leq 0 \ .$$

¿From this choice, it follows, of course, that the segment joining y_0 and z meets the hyperplane $(A(x))^{-1}(\frac{\alpha(x)}{r})$. In other words, there is $\lambda \in [0, 1]$ such that

$$A(x)(\lambda z + (1-\lambda)y_0) = \frac{\alpha(x)}{r} \ . \tag{2}$$

So, if we put $y = \lambda z + (1-\lambda)y_0$, we have $y \in F(x)$ and

$$\|y - y_0\| = \lambda\|z - y_0\| \ . \tag{3}$$

We claim that $\|y - y_0\| < \epsilon$. This follows at once from (3) if $\lambda < \frac{\epsilon}{2}$. Thus, assume $\lambda \geq \frac{\epsilon}{2}$. In this case, to prove our claim, it is enough to show that

$$2(1-\delta) < \|z + y_0\| \tag{4}$$

since (4) implies $\|z - y_0\| < \epsilon$. To this end, note that, by (2), one has

$$\frac{\left|A(x)(y_0) - \frac{\alpha(x)}{r}\right|}{\|A(x)\|_{E^*}} = \frac{\lambda|A(x)(z - y_0)|}{\|A(x)\|_{E^*}} \ ,$$

and so, from (1), it follows that

$$\frac{|A(x)(z - y_0)|}{\|A(x)\|_{E^*}} < \delta \ . \tag{5}$$

Suppose $A(x)(z) = \|A(x)\|_{E^*}$. Then, from (5), we get

$$1 - \delta < \frac{A(x)(y_0)}{\|A(x)\|_{E^*}} \ . \tag{6}$$

On the other hand, we also have

$$1 + \frac{A(x)(y_0)}{\|A(x)|_{E^*}} = \frac{A(x)(z + y_0)}{\|A(x)\|_{E^*}} \leq \|z + y_0\| \ . \tag{7}$$

So, (4) follows from (6) and (7). Now, suppose $A(x)(z) = -\|A(x)\|_{E^*}$. Then, from (5), we get

$$1 - \delta < -\frac{A(x)(y_0)}{\|A(x)\|_{E^*}} \ . \tag{8}$$

On the other hand, we have

$$1 - \frac{A(x)(y_0)}{\|A(x)|_{E^*}} = -\frac{A(x)(z + y_0)}{\|A(x)\|_{E^*}} \leq \|z + y_0\| \ . \tag{9}$$

So, in the present case, (4) is a consequence of (8) and (9). In such a manner, we have proved that F is lower semicontinuous at x_0. Hence, it remains proved that F is lower semicontinuous in X with respect to the strong topology, and so, *a fortiori*, with respect to τ. Since F is also with nonempty τ-closed convex values, and $\frac{g}{r}$ is a τ-continuous selection of it over the closed set C, by Theorem A, F admits a τ-continuous selection ω in X such that $\omega_{|C} = \frac{g}{r}$. At this point, if we put $\psi = r\omega$, it follows that ψ is a τ-continuous function, from X into B_r, such that $\psi_{|C} = g$ and $A(x)(\psi(x)) = \alpha(x)$ for all $x \in X$, against the hypotheses. This concludes the proof. $\triangle$

We now indicate two reasonable ways of application of Theorem 1. The first one is based on the Tychonoff fixed point theorem.

THEOREM 2. - *Assume that E is a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the weak to the strong topology. Assume that there exist a weakly continuous function $\alpha : B_r \to \mathbf{R}$ satisfying $|\alpha(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in B_r$, and a weakly continuous function $g : C \to B_r$ such*

$$\langle A(x), g(x) \rangle = \alpha(x) \ \text{ and } \ g(x) \neq x$$

for all $x \in C$, where

$$C = \{x \in B_r : \langle A(x), x \rangle = \alpha(x)\} \ .$$

Then, there exists $x^* \in B_r$ such that $A(x^*) = 0$.

PROOF. Identifying E with E^*, we apply Theorem 1 taking $X = B_r$, with the relativization of the weak topology of E, and taking as τ the weak topology of E. Due to the kind of continuity we are assuming for A, the function $x \to \langle A(x), x \rangle$ turns out to be weakly continuous (see the proof of Theorem 4), and so the set C is weakly closed. Now, let $\psi : B_r \to B_r$ be any weakly continuous function such that $\psi_{|C} = g$. By the Tychonoff fixed point theorem, there is $x_0 \in B_r$ such that $\psi(x_0) = x_0$. Since g ha no fixed points in C, it follows that $x_0 \notin C$, and so

$$\langle A(x_0), \psi(x_0) \rangle = \langle A(x_0), x_0 \rangle \neq \alpha(x_0) \ .$$

Hence, all the assumptions of Theorem 1 are satisfied, and the conclusion follows from it. $\triangle$

It is worth noticing the following consequence of Theorem 2.

THEOREM 3. - *Let E and A be as in Theorem 2. Assume that for each $x \in B_r$, with $\|A(x)\| > r$, one has*

$$\left\| A\left(\frac{rA(x)}{\|A(x)\|}\right) \right\| \leq r \ . \tag{10}$$

Then, the operator A has either a zero or a fixed point.

PROOF. Define the function $\alpha : B_r \to \mathbf{R}$ by

$$\alpha(x) = \begin{cases} \|A(x)\|^2 & \text{if } \|A(x)\| \leq r \\ r\|A(x)\| & \text{if } \|A(x)\| > r \ . \end{cases}$$

Clearly, the function α is weakly continuous and satisfies $|\alpha(x)| \leq r\|A(x)\|$ for all $x \in B_r$. Put

$$C = \{x \in B_r : \langle A(x), x\rangle = \alpha(x)\} \ .$$

Note that if $x \in C$ then $\|A(x)\| \leq r$. Indeed, otherwise, we would have $\langle A(x), x\rangle = r\|A(x)\|$, and so, necessarily, $x = \frac{rA(x)}{\|A(x)\|}$, against (10). Hence, we have $\langle A(x), A(x)\rangle = \alpha(x)$ for all $x \in C$. At this point, the conclusion follows at once from Theorem 2, taking $g = A_{|C}$. $\triangle$

REMARK 1. - It would be interesting to know whether Theorem 3 can be improved assuming that A is a continuous operator with relatively compact range.

The second application of Theorem 1 is based on connectedness arguments. For other results of this type we refer to [5] (see also [2]).

THEOREM 4. - *Let X be a connected paracompact topological space and $A : X \to E^*$ a weakly continuous and locally bounded operator. Assume that there exist $r > 0$, a closed set $C \subset X$, a continuous function $g : C \to B_r$ and an upper semicontinuous function $\beta : X \to \mathbf{R}$, with $|\beta(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in X$, such that $g(C)$ is disconnected,*

$$\beta(x) \leq A(x)(g(x))$$

for all $x \in C$ and

$$A(x)(y) < \beta(x)$$

for all $x \in X \setminus C$ and $y \in B_r \setminus g(C)$.

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

PROOF. First, note that the function $x \to A(x)(g(x))$ is continuous in C. To see this, let $x_1 \in C$ and let $\{x_\gamma\}_{\gamma\in D}$ be any net in C converging to x_1. By assumption, there are $M > 0$ and a neighbourhood U of x_1 such that $\|A(x)\|_{E^*} \leq M$ for all $x \in U$. Let $\gamma_0 \in D$ be such that $x_\gamma \in U$ for all $\gamma \geq \gamma_0$. Thus, for each $\gamma \geq \gamma_0$, one has

$$|A(x_\gamma)(g(x_\gamma)) - A(x_1)(g(x_1))| \leq M\|g(x_\gamma) - g(x_1)\| + |A(x_\gamma)(g(x_1)) - A(x_1)(g(x_1))|$$

from which, of course, it follows that $\lim_\gamma A(x_\gamma)(g(x_\gamma)) = A(x_1)(g(x_1))$. Next, observe that the multifunction $x \to [\beta(x), r\|A(x)\|_{E^*}]$ is lower semicontinuous and that the function $x \to A(x)(g(x))$ is a continuous selection of it in C. Hence, by Michael's theorem, there is a continuous function $\alpha : X \to \mathbf{R}$ such that $\alpha(x) = A(x)(g(x))$ for all $x \in C$ and $\beta(x) \leq \alpha(x) \leq r\|A(x)\|_{E^*}$ for all $x \in X$. Now, let $\psi : X \to B_r$ be any continuous function such that $\psi_{|C} = g$. Since X is connected, $\psi(X)$ is connected too. But then, since $g(C)$ is disconnected and $g(C) \subset \psi(X)$, there exists $y_0 \in \psi(X) \setminus g(C)$. Let $x_0 \in X \setminus C$ be such that $\psi(x_0) = y_0$. So, by hypothesis, we have

$$A(x_0)(\psi(x_0)) = A(x_0)(y_0) < \beta(x_0) \leq \alpha(x_0) \ .$$

Hence, taking as τ the strong topology of E, all the assumptions of Theorem 1 are satisfied, and the conclusion follows from it. $\triangle$

REMARK 2. - Observe that when X is first-countable, the local boundedness of A follows automatically from its weak continuity. This follows from the fact that, in a Banach space, any weakly convergent sequence is bounded.

It is worth noticing the corollary of Theorem 4 which comes out taking $X = B_r$, $\beta = 0$ and g=identity:

THEOREM 5. - *Let E be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the strong to the weak topology. Assume that the set $C = \{x \in B_r : \langle A(x), x \rangle \geq 0\}$ is disconnected and that, for each $x, y \in B_r \setminus C$, one has $\langle A(x), y \rangle < 0$.*

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

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